

# ON THE FIXED POINT PROPERTY IN DIRECT SUMS OF BANACH SPACES WITH STRICTLY MONOTONE NORMS

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**ABSTRACT.** It is shown that if a Banach space  $X$  has the weak Banach–Saks property and the weak fixed point property for nonexpansive mappings and  $Y$  satisfies property asymptotic (P) (which is weaker than the condition  $WCS(Y) > 1$ ), then  $X \oplus Y$  endowed with a strictly monotone norm enjoys the weak fixed point property. The same conclusion is valid if  $X$  admits a 1-unconditional basis.

## 1. INTRODUCTION

One of the classic problems of metric fixed point theory concerns existence of fixed points of nonexpansive mappings. Let  $C$  be a nonempty bounded closed and convex subset of a Banach space  $X$ . A mapping  $T : C \rightarrow C$  is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all  $x, y \in C$ . A Banach space  $X$  is said to have the fixed point property (FPP) if every such mapping has a fixed point. Adding the assumption that  $C$  is weakly compact in this condition, we obtain the definition of the weak fixed point property (WFPP).

In 1965, F. Browder [8] proved that Hilbert spaces have FPP. In the same year, Browder [9] and D. Göhde [27] showed independently that uniformly convex spaces have FPP, and W. A. Kirk [32] proved a more general result stating that all Banach spaces with weak normal structure have WFPP. Recall that a Banach space  $X$  has weak normal structure if  $r(C) < \text{diam } C$  for all weakly compact convex subsets  $C$  of  $X$  consisting of more than one point, where  $r(C) = \inf_{x \in C} \sup_{y \in C} \|x - y\|$  is the Chebyshev radius of  $C$ . There have been numerous discoveries since then. In 1981, D. Alspach [2] showed an example of a nonexpansive self-mapping defined on a weakly compact convex subset of  $L_1[0, 1]$  without a fixed point, and B. Maurey [43] used the Banach space ultraproduct construction to prove FPP for all reflexive subspaces of  $L_1[0, 1]$  as well as WFPP for  $c_0$ , see also [19]. Maurey's method was applied by P.-K. Lin [38] who proved that every Banach space with a 1-unconditional basis enjoys WFPP. In 1997, P. Dowling and C. Lennard [17] proved that every nonreflexive subspace of  $L_1[0, 1]$  fails FPP and they developed their techniques in the series of papers. For a fuller discussion of metric fixed point theory we refer the reader to [3, 26, 28, 31].

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Major progress in fixed point problems for nonexpansive mappings has been made recently. In 2003 (published in 2006), J. García Falset, E. Lloréns Fuster and E. Mazcuñan Navarro [24], (see also [45]), solved a long-standing problem in the theory by proving FPP for all uniformly nonsquare Banach spaces. In 2004, Dowling, Lennard and Turett [18] proved that a nonempty closed bounded convex subset of  $c_0$  has FPP if and only if it is weakly compact. In a recent paper [39], P.-K. Lin showed that a certain renorming of  $\ell_1$  enjoys FPP, thus solving another long-standing problem (FPP does not imply reflexivity).

The problem of whether FPP or WFPP is preserved under direct sums of Banach spaces has been thoroughly studied since the 1968 Belluce–Kirk–Steiner theorem [7], which states that a direct sum of two Banach spaces with normal structure, endowed with the maximum norm, also has normal structure. In 1984, T. Landes [36] showed that normal structure is preserved under a large class of direct sums including all  $\ell_p^N$ -sums,  $1 < p \leq \infty$ , but not under  $\ell_1^N$ -direct sums (see [37]). In 1999, B. Sims and M. Smyth [51] proved that both property (P) and asymptotic (P) are preserved under finite direct sums with monotone norms, see Section 2 for the relevant definitions. Nowadays, there are many results concerning permanence properties of normal structure and conditions which imply normal structure (see [16, 51]), but only few papers treat a general case of permanence of FPP, see [14, 34, 42, 54] and references therein.

In Section 3 we prove two quite general fixed point theorems for direct sums. Theorem 1 states that if  $X$  has the weak Banach–Saks property and WFPP, and  $Y$  has property asymptotic (P), then  $X \oplus Y$ , endowed with a strictly monotone norm, has WFPP. This is a strong extension of the second named author’s results [54]. A combination of the arguments contained in the proof of Theorem 1 with the ideas of P.-K. Lin [38] enables us to obtain in Theorem 2 the same conclusion if  $X$  has a 1-unconditional basis, see also a remark at the end of the paper.

## 2. PRELIMINARIES

Let  $X$  be a Banach space and  $D \subset X$  be a nonempty set. Given  $r > 0$ , we put

$$B(D, r) = \{x \in X : \|x - y\| \leq r \text{ for some } y \in D\}.$$

If  $D = \{x_0\}$  for some  $x_0 \in X$ , then this is just the closed ball  $B(x_0, r)$ .

The following construction is crucial for many existence fixed point theorems for nonexpansive mappings. Assume that there exists a nonexpansive mapping  $T : C \rightarrow C$  without a fixed point, where  $C$  is a nonempty weakly compact convex subset of a Banach space  $X$ . Then, by the Kuratowski–Zorn lemma, we obtain a convex and weakly compact set  $K \subset C$  which is minimal invariant under  $T$  and which is not a singleton. It follows from the Banach contraction principle that  $K$  contains an approximate fixed point sequence  $(x_n)$  for  $T$ , i.e.,

$$\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0.$$

The following lemma was proved independently by K. Goebel [25] and L. Karlovitz [29].

**Lemma 1.** *Let  $K$  be a minimal invariant set for a nonexpansive mapping  $T$ . If  $(x_n)$  is an approximate fixed point sequence for  $T$  in  $K$ , then*

$$\lim_{n \rightarrow \infty} \|x_n - x\| = \text{diam } K$$

for every  $x \in K$ .

The above lemma can be reformulated in terms of Banach space ultraproducts as follows (see, e.g., [1, 49]). Let  $\mathcal{U}$  be a free ultrafilter on  $\mathbb{N}$ . The ultrapower  $\tilde{X} := (X)_{\mathcal{U}}$  of a Banach space  $X$  is the quotient space of

$$l_{\infty}(X) = \left\{ (x_n) : x_n \in X \text{ for all } n \in \mathbb{N} \text{ and } \|(x_n)\| = \sup_n \|x_n\| < \infty \right\}$$

by

$$\mathcal{N} = \left\{ (x_n) \in l_{\infty}(X) : \lim_{n \rightarrow \mathcal{U}} \|x_n\| = 0 \right\}.$$

Here  $\lim_{n \rightarrow \mathcal{U}}$  denotes the ultralimit over  $\mathcal{U}$ . One can prove that the quotient norm on  $\tilde{X}$  is given by

$$\|(x_n)_{\mathcal{U}}\| = \lim_{n \rightarrow \mathcal{U}} \|x_n\|,$$

where  $(x_n)_{\mathcal{U}}$  is the equivalence class of  $(x_n)$ . It is also clear that  $X$  is isometric to a subspace of  $\tilde{X}$  by the mapping  $x \mapsto (x, x, \dots)_{\mathcal{U}}$ . We shall not distinguish between  $x$  and  $(x, x, \dots)_{\mathcal{U}}$ . Let

$$\tilde{K} = \left\{ (x_n)_{\mathcal{U}} \in \tilde{X} : x_n \in K \text{ for all } n \in \mathbb{N} \right\}.$$

We extend the mapping  $T$  to  $\tilde{K}$  by setting  $\tilde{T}((x_n)_{\mathcal{U}}) = (Tx_n)_{\mathcal{U}}$ . It is not difficult to see that  $\tilde{T} : \tilde{K} \rightarrow \tilde{K}$  is a well-defined nonexpansive mapping. Moreover, the set  $\text{Fix } \tilde{T}$  of fixed points of  $\tilde{T}$  is nonempty and consists of all those points in  $\tilde{K}$  which are represented by sequences  $(x_n)$  in  $K$  for which  $\lim_{n \rightarrow \mathcal{U}} \|Tx_n - x_n\| = 0$ . It follows from the Goebel–Karlovitz Lemma 1 ( $K$  is minimal invariant) that

$$(1) \quad \|x - \tilde{y}\| = \text{diam } K$$

for every  $x \in K$  and  $\tilde{y} \in \text{Fix } \tilde{T}$ . Even more can be said.

**Lemma 2** (see Lin [38]). *Let  $K$  be a minimal invariant set for a nonexpansive mapping  $T$ . If  $(\tilde{u}_k)$  is an approximate fixed point sequence for  $\tilde{T}$  in  $\tilde{K}$ , then*

$$\lim_{k \rightarrow \infty} \|\tilde{u}_k - x\| = \text{diam } K$$

for every  $x \in K$ .

We conclude this section with recalling several properties of a Banach space  $X$  which are sufficient for weak normal structure. Let

$$N(X) = \inf \left\{ \frac{\text{diam } A}{r(A)} \right\},$$

where the infimum is taken over all bounded convex sets  $A \subset X$  with  $\text{diam } A > 0$ . Assuming that  $X$  does not have the Schur property, we put

$$WCS(X) = \inf \left\{ \frac{\text{diam}_a(x_n)}{r_a(x_n)} \right\},$$

where the infimum is taken over all sequences  $(x_n)$  which converge to 0 weakly but not in norm, see [10]. Here

$$\text{diam}_a(x_n) = \lim_{n \rightarrow \infty} \sup_{k, l \geq n} \|x_k - x_l\|$$

denotes the asymptotic diameter of  $(x_n)$  and

$$r_a(x_n) = \inf \left\{ \limsup_{n \rightarrow \infty} \|x_n - x\| : x \in \overline{\text{conv}}(x_n)_{n=1}^\infty \right\}$$

denotes the asymptotic radius of  $(x_n)$ .

We say that a Banach space  $X$  has uniform normal structure if  $N(X) > 1$  and weak uniform normal structure (or satisfies Bynum's condition) if  $WCS(X) > 1$ . Moreover,  $X$  is said to have property (P) if

$$\liminf_{n \rightarrow \infty} \|x_n\| < \text{diam}(x_n)_{n=1}^\infty$$

whenever  $(x_n)$  converges weakly to 0 and  $\text{diam}(x_n)_{n=1}^\infty > 0$ , see [52], and  $X$  has property asymptotic (P) if

$$\liminf_{n \rightarrow \infty} \|x_n\| < \text{diam}_a(x_n)$$

whenever  $(x_n)$  converges weakly to 0 and  $\text{diam}_a(x_n) > 0$ , see [50]. It is known (see, e.g., [51]) that

$N(X) > 1 \Rightarrow WCS(X) > 1 \Rightarrow \text{asymptotic (P)} \Rightarrow \text{(P)} \Rightarrow \text{weak normal structure}$ .

### 3. RESULTS

In the sequel we shall need the following result (see [15, 51]).

**Lemma 3.** *Every bounded sequence  $(x_n)$  in a Banach space  $X$  contains a subsequence  $(y_n)$  such that the following limit exists*

$$\lim_{n, m \rightarrow \infty, n \neq m} \|y_n - y_m\|.$$

Let us now recall terminology concerning direct sums. A norm  $\|\cdot\|$  on  $\mathbb{R}^2$  is said to be monotone if

$$\|(x_1, y_1)\| \leq \|(x_2, y_2)\|$$

whenever  $0 \leq x_1 \leq x_2$ ,  $0 \leq y_1 \leq y_2$ . A norm  $\|\cdot\|$  is said to be strictly monotone if

$$\|(x_1, y_1)\| < \|(x_2, y_2)\|$$

whenever  $0 \leq x_1 \leq x_2$ ,  $0 \leq y_1 < y_2$  or  $0 \leq x_1 < x_2$ ,  $0 \leq y_1 \leq y_2$ . It is easy to see that  $\ell_p^2$ -norms,  $1 \leq p < \infty$ , are strictly monotone.

We shall tacitly assume that

$$(2) \quad \|(1, 0)\| = 1 = \|(0, 1)\|.$$

This does not result in loss of generality because given a strictly monotone norm  $\|\cdot\|$  on  $\mathbb{R}^2$ , we can find another strictly monotone norm  $\|\cdot\|_1$  such that the spaces  $(\mathbb{R}^2, \|\cdot\|_1)$ ,  $(\mathbb{R}^2, \|\cdot\|)$  are isometric and  $\|\cdot\|_1$  satisfies (2). Moreover, all conditions appearing in our results are isometric invariant.

Let  $Z$  be a normed space  $(\mathbb{R}^2, \|\cdot\|_Z)$ . We shall write  $X \oplus_Z Y$  for the  $Z$ -direct sum of Banach spaces  $X, Y$  with the norm  $\|(x, y)\| = \|(\|x\|, \|y\|)\|_Z$ , where  $(x, y) \in X \times Y$ .

**Lemma 4.** *Let  $X \oplus_Z Y$  be a direct sum of Banach spaces  $X, Y$  with respect to a strictly monotone norm. Assume that  $Y$  has property asymptotic (P), the vectors  $v_n = (x_n, y_n) \in X \oplus_Z Y$  tend weakly to 0 and*

$$\lim_{n, m \rightarrow \infty, n \neq m} \|v_n - v_m\| = \lim_{n \rightarrow \infty} \|v_n\|.$$

Then  $\lim_{n \rightarrow \infty} \|y_n\| = 0$ .

*Proof.* Suppose that the sequence  $(y_n)$  does not converge to 0. Then we can assume that the following limits exist

$$\lim_{n \rightarrow \infty} \|x_n\|, \quad \lim_{n, m \rightarrow \infty, n \neq m} \|x_n - x_m\|, \quad \lim_{n \rightarrow \infty} \|y_n\|, \quad \lim_{n, m \rightarrow \infty, n \neq m} \|y_n - y_m\|$$

(see Lemma 3) and  $\lim_{n \rightarrow \infty} \|y_n\| > 0$ . The sequence  $(x_n)$  and  $(y_n)$  converges weakly to 0 in  $X$  and  $Y$ , respectively. It follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n\| &\leq \lim_{n, m \rightarrow \infty, n \neq m} \|x_n - x_m\|, \\ \lim_{n \rightarrow \infty} \|y_n\| &\leq \lim_{n, m \rightarrow \infty, n \neq m} \|y_n - y_m\|. \end{aligned}$$

Hence

$$\begin{aligned} \lim_{n \rightarrow \infty} \|v_n\| &= \|(\lim_{n \rightarrow \infty} \|x_n\|, \lim_{n \rightarrow \infty} \|y_n\|)\|_Z \\ &\leq \|(\lim_{n, m \rightarrow \infty, n \neq m} \|x_n - x_m\|, \lim_{n, m \rightarrow \infty, n \neq m} \|y_n - y_m\|)\|_Z \\ &= \lim_{n, m \rightarrow \infty, n \neq m} \|v_n - v_m\| = \lim_{n \rightarrow \infty} \|v_n\|. \end{aligned}$$

The norm  $\|\cdot\|_Z$  is strictly monotone, so

$$\lim_{n \rightarrow \infty} \|y_n\| = \lim_{n, m \rightarrow \infty, n \neq m} \|y_n - y_m\| = \text{diam}_a(y_n)$$

which contradicts our assumption that  $Y$  has property asymptotic (P).  $\square$

Let  $X$  be a Banach space and  $x, y \in X$ . By the metric segment with the endpoints  $x, y$  we mean the set

$$S(x, y) = \{z \in X : \|x - z\| + \|z - y\| = \|x - y\|\}.$$

Clearly,  $S(x, y)$  contains the algebraic segment  $\text{conv}\{x, y\}$ .

**Lemma 5.** *Let  $X \oplus_Z Y$  be a direct sum of Banach spaces  $X, Y$  with respect to a strictly monotone norm and  $\mathcal{U}$  be a free ultrafilter on  $\mathbb{N}$ . Let  $X_0$  denote the set of all elements of  $(X \oplus_Z Y)_{\mathcal{U}}$  of the form  $((x_n, 0))_{\mathcal{U}}$  where  $(x_n) \in \ell_{\infty}(X)$ . If  $u, v \in X_0$ , then  $S(u, v) \subset X_0$ .*

*Proof.* Let  $u = ((x_n, 0))_{\mathcal{U}}$ ,  $v = ((y_n, 0))_{\mathcal{U}}$  and  $z = ((a_n, b_n))_{\mathcal{U}}$  where  $(x_n)$ ,  $(y_n)$ ,  $(a_n) \in \ell_\infty(X)$ ,  $(b_n) \in \ell_\infty(Y)$ . Assume that  $\lim_{n \rightarrow \mathcal{U}} \|b_n\| > 0$  and  $z \in S(u, v)$ . Since the norm  $\|\cdot\|_Z$  is strictly monotone,

$$\begin{aligned} \lim_{n \rightarrow \mathcal{U}} \|x_n - y_n\| &= \|u - v\| = \|u - z\| + \|z - v\| \\ &= \|(\lim_{n \rightarrow \mathcal{U}} \|x_n - a_n\|, \lim_{n \rightarrow \mathcal{U}} \|b_n\|)\|_Z + \|(\lim_{n \rightarrow \mathcal{U}} \|y_n - a_n\|, \lim_{n \rightarrow \mathcal{U}} \|b_n\|)\|_Z \\ &> \|(\lim_{n \rightarrow \mathcal{U}} \|x_n - a_n\|, 0)\|_Z + \|(\lim_{n \rightarrow \mathcal{U}} \|y_n - a_n\|, 0)\|_Z \\ &= \lim_{n \rightarrow \mathcal{U}} \|x_n - a_n\| + \lim_{n \rightarrow \mathcal{U}} \|a_n - y_n\| \geq \lim_{n \rightarrow \mathcal{U}} \|x_n - y_n\|. \end{aligned}$$

This contradiction shows that  $\lim_{n \rightarrow \mathcal{U}} \|b_n\| = 0$  and consequently,  $z = ((a_n, 0))_{\mathcal{U}}$ .  $\square$

Recall that a Banach space  $X$  is said to have the weak Banach–Saks property if each weakly null sequence  $(w_n)$  in  $X$  admits a subsequence  $(x_n)$  whose arithmetic means converge to 0 in norm, i.e.,

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{k=1}^n x_k \right\| = 0.$$

S. A. Rakov [48] proved a result which can be formulated in the following way (see also [20, 21]). If  $(w_n)$  is a weakly null sequence in a Banach space  $X$  with the weak Banach–Saks property, then there is a subsequence  $(x_n)$  of  $(w_n)$  such that

$$(3) \quad \lim_{m \rightarrow \infty} \sup \left\{ \left\| \frac{1}{m} \sum_{i=1}^m x_{p_i} \right\| : p_1 < p_2 < \cdots < p_m \right\} = 0.$$

In the proof of the next theorem we shall use the following well-known construction. Let  $C$  be a nonempty convex closed subset of a Banach space  $X$  and consider a continuous mapping  $T : C \rightarrow C$ . Given a separable subset  $D$  of  $C$ , we set  $C_1 = \text{conv } D$  and  $C_{n+1} = \text{conv}(C_n \cup T(C_n))$  for  $n \in \mathbb{N}$ . It is easy to see that the set

$$C(D) = \overline{\bigcup_{n \in \mathbb{N}} C_n}$$

is closed, convex, separable and  $T$ -invariant. Actually,  $C(D)$  is the smallest closed convex  $T$ -invariant set containing  $D$ .

**Theorem 1.** *Let  $X$  be a Banach space with the weak Banach–Saks property and WFPP. If  $Y$  has property asymptotic (P), then  $X \oplus_Z Y$ , endowed with a strictly monotone norm, has WFPP.*

*Proof.* Assume that  $X \oplus_Z Y$  does not have WFPP. Then, there exists a weakly compact convex subset  $C$  of  $X \oplus_Z Y$  and a nonexpansive mapping  $T : C \rightarrow C$  without a fixed point. By the standard argument described in Section 2, there exists a convex and weakly compact set  $K \subset C$  which is minimal invariant under  $T$ . Let  $(w_n)$  be an approximate fixed point sequence for  $T$  in  $K$ . Without loss of generality we can assume that  $\text{diam } K = 1$  and  $(w_n)$  converges weakly to  $(0, 0) \in K$ . In view of Lemma 3 we can assume

that the double limit  $\lim_{n,m \rightarrow \infty, n \neq m} \|w_n - w_m\|$  exists. From Lemma 1 it follows that

$$(4) \quad \lim_{n,m \rightarrow \infty, n \neq m} \|w_n - w_m\| = 1 = \lim_{n \rightarrow \infty} \|w_n\|.$$

Since  $X$  has the weak Banach–Saks property, we can find a subsequence  $(x_n)$  of  $(w_n)$  for which condition (3) holds.

We shall construct by induction a sequence  $(n_k^1), (n_k^2), \dots$  of increasing sequences of natural numbers and an ascending sequence  $(D_n)$  of subsets of  $\tilde{K}$  such that for every  $m \in \mathbb{N}$  the following conditions hold

- (i) the set  $\mathbb{N} \setminus \bigcup_{i=1}^m A_i$  is infinite and contains  $A_{m+1}$  where  $A_i = \{n_k^i : k \in \mathbb{N}\}$ ,
- (ii)  $D_1 = \{v_1\}$  and  $D_{m+1} \subset \overline{\bigcup_{y \in D_m} S(y, v_{m+1})}$ ,
- (iii)  $D_m$  is closed, convex, separable,  $\tilde{T}$ -invariant and  $\text{conv}\{v_1, \dots, v_m\} \subset D_m$

where  $v_i = (x_{n_k^i})_{\mathcal{U}}$  for every  $i \in \mathbb{N}$ .

To this end we put  $n_k^1 = 2k - 1$  for every  $k \in \mathbb{N}$ . Suppose now that we have desirable sequences  $(n_k^1), \dots, (n_k^m)$  and sets  $D_1, \dots, D_m$ . Then the set  $A = \mathbb{N} \setminus \bigcup_{i=1}^m A_i$  is infinite. Let  $\{u_n : n \in \mathbb{N}\}$  be a dense subset of  $D_m$ . We have  $u_k = (y_n^k)_{\mathcal{U}}$  for some sequence  $(y_n^k)$  in  $K$ . Using Lemma 1, for every  $k \in \mathbb{N}$  we find  $n_k^{m+1} \in A$  so that  $\|y_k^i - x_{n_k^{m+1}}\| \geq 1 - \frac{1}{2^k}$  for  $i = 1, \dots, k$ , the sequence  $(n_k^{m+1})$  is increasing and the set  $A \setminus A_{m+1}$  is infinite. We have

$$\|u_i - v_{m+1}\| = \lim_{k \rightarrow \mathcal{U}} \|y_k^i - x_{n_k^{m+1}}\| = 1$$

for every  $i \in \mathbb{N}$ . It clearly follows that  $\|u - v_{m+1}\| = 1$  for every  $u \in D_m$ .

We put  $D_{m+1} = C(D_m \cup \{v_{m+1}\})$ . To show that (ii) is satisfied observe that the set  $E = \bigcup_{y \in D_m} S(y, v_{m+1})$  is convex and  $\tilde{T}$ -invariant. Indeed, if  $u_1, u_2 \in E$ , then there are  $y_1, y_2 \in D_m$  such that

$$\|y_i - u_i\| + \|u_i - v_{m+1}\| = \|y_i - v_{m+1}\| = 1$$

for  $i = 1, 2$ . Given  $t \in [0, 1]$ , we have  $(1-t)y_1 + ty_2 \in D_m$  and therefore,

$$\begin{aligned} & \| (1-t)y_1 + ty_2 - ((1-t)u_1 + tu_2) \| + \| (1-t)u_1 + tu_2 - v_{m+1} \| \\ & \leq (1-t)(\|y_1 - u_1\| + \|y_1 - v_{m+1}\|) + t(\|y_2 - u_2\| + \|y_2 - v_{m+1}\|) \\ & = (1-t)\|y_1 - v_{m+1}\| + t\|y_2 - v_{m+1}\| = 1 \\ & = \| (1-t)y_1 + ty_2 - v_{m+1} \|. \end{aligned}$$

This shows that  $(1-t)u_1 + tu_2 \in E$ . Moreover,  $\tilde{T}v_{m+1} = v_{m+1}$  and  $\tilde{T}y_1 \in D_m$ , so

$$\begin{aligned} \|\tilde{T}y_1 - \tilde{T}u_1\| + \|\tilde{T}u_1 - v_{m+1}\| & \leq \|y_1 - u_1\| + \|u_1 - v_{m+1}\| = 1 \\ & = \|\tilde{T}y_1 - v_{m+1}\|. \end{aligned}$$

Therefore,  $\tilde{T}u_1 \in E$ . Consequently,  $E$  is convex,  $\tilde{T}$ -invariant and  $D_m \cup \{v_{m+1}\} \subset E$  which easily gives us condition (ii). Condition (iii) is obvious.

We put  $D = \overline{\bigcup_{m \in \mathbb{N}} D_m}$ . Then  $\frac{1}{m} (\sum_{i=1}^m v_i) \in D$  for every  $m \in \mathbb{N}$  and from (3) we see that

$$\lim_{m \rightarrow \infty} \left\| \frac{1}{m} \left( \sum_{i=1}^m v_i \right) \right\| = \lim_{m \rightarrow \infty} \lim_{k \rightarrow \mathcal{U}} \left\| \frac{1}{m} \left( \sum_{i=1}^m x_{n_k^i} \right) \right\| = 0.$$

This shows that  $(0, 0) \in D$  and consequently  $M = D \cap K \neq \emptyset$ . Clearly,  $M$  is closed, convex and  $\tilde{T}$ -invariant.

Let  $X_0$  denote the set of all elements of  $(X \oplus_Z Y)_{\mathcal{U}}$  of the form  $((z_n, 0))_{\mathcal{U}}$  where  $(z_n) \in \ell_{\infty}(X)$ . In view of (4) and Lemma 4,  $v_n \in X_0$  for every  $n \in \mathbb{N}$ . Using (ii) and Lemma 5, one can now easily show that  $D_n \subset X_0$  for every  $n \in \mathbb{N}$ . Hence  $D \subset X_0$  and consequently  $M \subset X_0$ . We can therefore identify  $M$  with a subset of  $X$ . Since  $X$  has WFPP,  $T$  has a fixed point in  $M$  which contradicts our assumption.  $\square$

*Remark.* The construction of the set  $D$  is partly inspired by the arguments in the corrigendum to [54]. The idea of using metric segments to obtain a  $T$ -invariant set appeared earlier in [5].

It is well known that all superreflexive spaces,  $c_0$ ,  $\ell_1$  as well as  $L_1[0, 1]$  have the weak Banach–Saks property (see, e.g., [13]). In metric fixed point theory, the following coefficient introduced by J. García-Falset [22] plays an important role. Given a Banach space  $X$ , we put

$$R(X) = \sup \left\{ \liminf_{n \rightarrow \infty} \|x_1 + x_n\| \right\},$$

where the supremum is taken over all weakly null sequences  $(x_n)$  in the unit ball  $B_X$ . If  $R(X) < 2$ , then  $X$  has the weak Banach–Saks property (see [22]) and WFPP ([23], see also [47]). For more details about the Banach–Saks property see also [6, 11, 35, 41] and references therein.

**Corollary 1.** *Let  $X$  be a Banach space with  $R(X) < 2$  and  $Y$  have weak uniform normal structure. Then  $X \oplus_Z Y$ , endowed with a strictly monotone norm, has WFPP.*

It has recently been proved in [24] (see also [45]) that all uniformly non-square Banach spaces have FPP. Also, uniformly noncreasy spaces introduced in [46] are superreflexive and have FPP. Other examples of superreflexive Banach spaces without normal structure but with FPP are given by the results in [53, 54].

**Corollary 2.** *Let  $X$  be a uniformly nonsquare or uniformly noncreasy Banach space and let  $Y$  have weak uniform normal structure. Then  $X \oplus_Z Y$ , endowed with a strictly monotone norm, has WFPP.*

In our next result we deal with Banach spaces admitting 1-unconditional bases. Recall that a Schauder basis  $(e_n)$  of a Banach space  $X$  is said to be an unconditional basis provided that for every choice of signs  $(\epsilon_n)$ ,  $\epsilon_n = \pm 1$ , the series  $\sum \epsilon_n \alpha_n e_n$  converges whenever  $\sum \alpha_n e_n$  converges. Then the supremum

$$\lambda = \sup \left\{ \left\| \sum_{n=1}^{\infty} \epsilon_n \alpha_n e_n \right\| : \left\| \sum_{n=1}^{\infty} \alpha_n e_n \right\| = 1, \epsilon_n = \pm 1 \right\}$$



is finite and it is called the unconditional constant of  $(e_n)$  (see [40, p. 18]). In this case we say that the basis  $(e_n)$  is  $\lambda$ -unconditional. Given a nonempty set  $F \subset \mathbb{N}$ , we put

$$P_F x = \sum_{n \in F} \alpha_n e_n$$

where  $x = \sum_{n=1}^{\infty} \alpha_n e_n$ . Clearly,  $P_F$  is a linear projection and  $\|P_F x\| \leq \lambda \|x\|$  for every  $x \in X$ .

It is well known that  $c_0$ ,  $\ell_p$ ,  $1 \leq p < \infty$  have 1-unconditional bases and the same is true for the space  $X_\beta$ , which is  $\ell_2$  endowed with the norm

$$\|x\|_\beta = \max \{\|x\|_2, \beta \|x\|_\infty\}.$$

If we combine the arguments from the first part of the proof of Theorem 1 with the ideas of Lin [38] we obtain the following result.

**Theorem 2.** *Let  $X$  be a Banach space with a 1-unconditional basis and let  $Y$  have property asymptotic (P). Then  $X \oplus_Z Y$ , endowed with a strictly monotone norm, has WFPP.*

*Proof.* Assume that  $X \oplus_Z Y$  does not have WFPP. Then, there exists a weakly compact convex subset  $K$  of  $X \oplus_Z Y$  which is minimal invariant under a nonexpansive mapping  $T$ . Arguing as in the proof of Theorem 1, we can assume that an approximate fixed point sequence  $((x_n, y_n))$  for  $T$  converges weakly to  $(0, 0) \in K$  and that  $(y_n)$  converges strongly to 0. Passing to a subsequence, we can therefore assume that

$$\lim_{n \rightarrow \infty} \|x_n - x_{n+1}\| = \text{diam } K = 1.$$

We can now follow the argument from [38, Theorem 1]. Let  $(e_n)$  be the 1-unconditional basis of  $X$ . By passing to subsequences, we can assume that there exists a sequence  $(F_n)$  of intervals of  $\mathbb{N}$  such that  $\max F_n < \min F_{n+1}$  for every  $n \in \mathbb{N}$ ,

$$\lim_{n \rightarrow \infty} \|P_{F_n} x_n - x_n\| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|P_{F_n} x_{n+1}\| = \lim_{n \rightarrow \infty} \|P_{F_{n+1}} x_n\| = 0$$

where  $P_{F_k}$  are the projections associated to the basis  $(e_n)$ . Clearly,  $P_{F_n} \circ P_{F_m} = 0$  if  $n \neq m$ ,

$$\lim_{n \rightarrow \infty} \|P_{F_n} x_n\| = \lim_{n \rightarrow \infty} \|x_n\| = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|P_{F_n} x\| = 0$$

for every  $x \in X$ .

Let  $\mathcal{U}$  be a free ultrafilter on  $\mathbb{N}$  and define projections  $\tilde{P}, \tilde{Q}$  on  $\tilde{X}$  by

$$\tilde{P}(u_n)_{\mathcal{U}} = (P_{F_n} u_n)_{\mathcal{U}}, \quad \tilde{Q}(u_n)_{\mathcal{U}} = (P_{F_{n+1}} u_n)_{\mathcal{U}}.$$

Put  $\tilde{y} = (x_n)_{\mathcal{U}}$ ,  $\tilde{z} = (x_{n+1})_{\mathcal{U}}$ . Then

$$(5) \quad \tilde{P}\tilde{y} = \tilde{y}, \quad \tilde{Q}\tilde{z} = \tilde{z} \quad \text{and} \quad \tilde{P}\tilde{z} = \tilde{Q}\tilde{y} = \tilde{P}x = \tilde{Q}x = 0$$

for every  $x \in K$ . Since  $(e_n)$  is 1-unconditional,

$$\|\tilde{y} + \tilde{z}\| = \|\tilde{y} - \tilde{z}\| = 1.$$

Let  $\tilde{v}_1 = ((x_n, 0))_{\mathcal{U}}$ ,  $\tilde{v}_2 = ((x_{n+1}, 0))_{\mathcal{U}}$  and

$$D = B(\tilde{v}_1, \frac{1}{2}) \cap B(\tilde{v}_2, \frac{1}{2}) \cap B(K, \frac{1}{2}) \cap \tilde{K}.$$

Note that  $D \neq \emptyset$  because  $\frac{1}{2}(\tilde{v}_1 + \tilde{v}_2) \in D$ . Moreover  $D$  is closed, convex and  $\tilde{T}(D) \subset D$ . Hence, there exists an approximate fixed point sequence for  $\tilde{T}$  in  $D$ . Fix an element  $((u_n, w_n))_{\mathcal{U}} \in D$ . The set  $D$  is contained in the metric segment  $S(\tilde{v}_1, \tilde{v}_2)$ . Lemma 5 shows therefore that  $((u_n, w_n))_{\mathcal{U}} = ((u_n, 0))_{\mathcal{U}}$ . Moreover, from the definition of  $D$ , there exists  $(x, y) \in K$  such that

$$\|\tilde{u} - x\| \leq \|((u_n, 0))_{\mathcal{U}} - (x, y)\| \leq \frac{1}{2}$$

where  $\tilde{u} = (u_n)_{\mathcal{U}}$ . Hence, with use of (5), we obtain

$$\begin{aligned} \|((u_n, w_n))_{\mathcal{U}}\| &= \|((u_n, 0))_{\mathcal{U}}\| = \|\tilde{u}\| \\ &= \frac{1}{2} \left\| \left( \tilde{u} - \tilde{P}\tilde{u} \right) + \left( \tilde{u} - \tilde{Q}\tilde{u} \right) + \left( \tilde{P}\tilde{u} + \tilde{Q}\tilde{u} \right) \right\| \\ &\leq \frac{1}{2} \left( \left\| (I - \tilde{P})(\tilde{u} - \tilde{y}) \right\| + \left\| (I - \tilde{Q})(\tilde{u} - \tilde{z}) \right\| + \left\| (\tilde{P} + \tilde{Q})(\tilde{u} - x) \right\| \right) \\ &\leq \frac{1}{2} (\|\tilde{u} - \tilde{y}\| + \|\tilde{u} - \tilde{z}\| + \|\tilde{u} - x\|) \leq \frac{3}{4}, \end{aligned}$$

which contradicts Lemma 2.  $\square$

*Remark.* In fact, just as in the proof of [38, Theorem 2], the argument works if  $X$  has an unconditional basis with the unconditional constant  $\lambda < (\sqrt{33} - 3)/2$ . Also, we can adopt the reasoning of M. A. Khamsi [30] (see also [1, Theorem 4.1]) to obtain the same conclusion if  $X$  is the James quasi-reflexive space.

Let us recall that there is a separable uniformly convex space which does not embed into a space with an unconditional basis (see [44]) and there is a Banach space with a 1-unconditional basis which does not have the weak Banach–Saks property (see [4, 12]). This shows that Theorems 1 and 2 are entirely independent of each other.

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